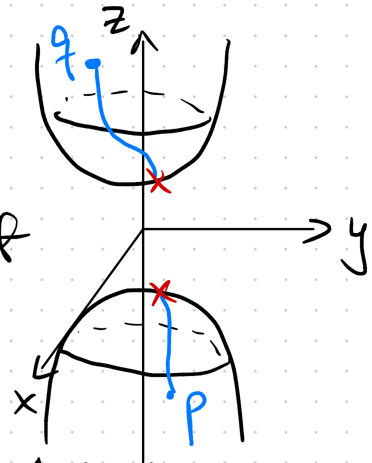
28/9/23

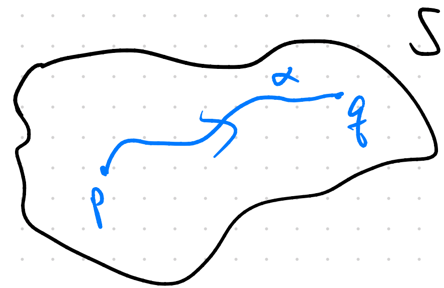
MATH 4030 Tutorial.

Example in lecture of a regular surface that is not connected.



Hyperboloid of 2 sheets.

Def: a surface $S \subseteq \mathbb{R}^3$ is connected if any two points on S can be joined by a continuous curve in S.



Q1: If $f: S \to \mathbb{R}$ is a nonzero continuous function on a connected surface S, then f does not change sign on S.

never 0

Pf: let $p, q \in S$, s.t. $f(p) > 0$, $f(q) < 0$. Since S is connected, $\exists$ cts curve $\alpha: [0,1] \to S$ s.t. $\alpha(0) = p$, $\alpha(1) = q$.

Consider $f \circ \alpha : [0,1] \to \mathbb{R}$. it is cts b/c α, f are cts.

$f \circ \alpha(0) = f(\alpha(0)) = f(p) > 0$

$f \circ \alpha(1) = f(q) < 0.$

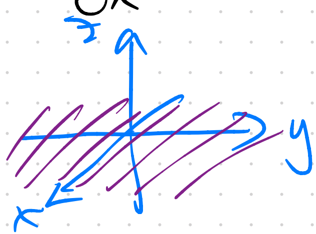
By intermediate value thm, $\exists\, c \in [0,1]$ s.t. $f \circ \alpha(c) = 0.$

$\Rightarrow f = 0$ at $\alpha(c) \in S.$ $\lightning$ //.

<u>Q2</u>: Show $f(x,y,z) = z^2$ has 0 <u>not</u> a regular value of f, but $f^{-1}(0)$ is a regular surface.

<u>Pf</u>: Clearly $f^{-1}(0)$ is the xy plane.

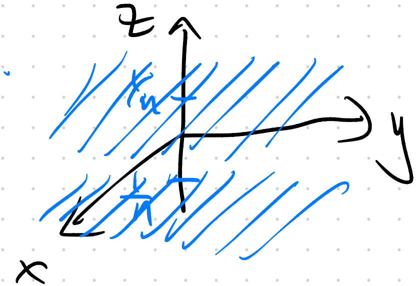
$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 2z.$$



$\Rightarrow df$ vanishes at $(0,0,0)$

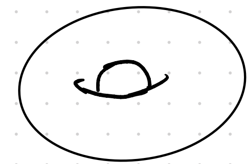
$\Rightarrow 0$ is a critical pt. of f and $f(0) = \underline{0}$ is <u>not</u> a regular value of f.

Problem above: multiplicity 2.



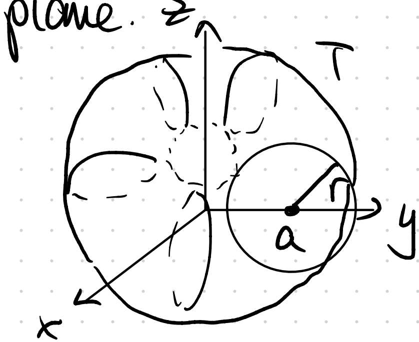
$f(x,y,z) = z.$

does not have 0 as a critical value since $f(0) = 0$, $\frac{\partial f}{\partial z} \equiv 1$, so $df(0) \neq 0$.



Q3: Show that the torus T, generated by rotating a circle of radius r about an axis at fixed distance $a > r$ is a regular surface.

Pf: let C be the circle of radius r, WLOG in the yz plane.



C is given by the equation

$$(y-a)^2 + z^2 = r^2$$

Rotating about z axis gives that points on T

satisfy: $z^2 + (\sqrt{x^2+y^2} - a)^2 = r^2$.

Let $f(x,y,z) = z^2 + (\sqrt{x^2+y^2} - a)^2$. So $T = f^{-1}(r^2)$,

so I WTS, r^2 is a regular value of f.

$$\frac{\partial f}{\partial x} = \cancel{2}(\sqrt{x^2+y^2} - a)\cancel{\tfrac{1}{2}}(x^2+y^2)^{-1/2} \cdot 2x = \frac{2x(\sqrt{x^2+y^2} - a)}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{2y(\sqrt{x^2+y^2} - a)}{\sqrt{x^2+y^2}}, \quad \frac{\partial f}{\partial z} = 2z.$$

f is differentiable as long as $(x,y) \neq (0,0)$.

and df vanishes only when

$$z = 0, \ x = 0 = y$$

$$z = 0, \ \sqrt{x^2+y^2} = a.$$

Since $a > r$, none of these points are in $f^{-1}(r^2)$

$$f(0, a\sin\theta, a\cos\theta) = 0^2 + \left(\sqrt{a^2\sin^2\theta + a^2\cos^2\theta} - a\right)^2 = 0 \neq r^2.$$
$$\theta \in [0, 2\pi)$$

So r^2 is a regular value of f.

Alternatively, param. T by

$$X(u,v) = ((r\cos u + a)\cos v, (r\cos u + a)\sin v, r\sin u)$$
$$0 < u < 2\pi, \quad 0 < v < 2\pi.$$

and show that we can cover T by coord. charts and

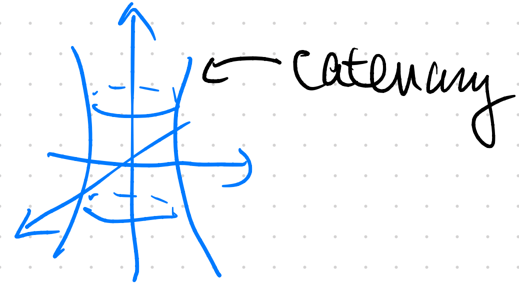
1) X is smooth

2) dX is full-rank

3) X is a homeomorphism between T and $\mathbb{R}^2$.

The torus is an example of a surface of revolution.

other examples: catenoid.



cylinder

These are given by the param.

$$X(u,v) = \big(f(u)\cos v,\ f(u)\sin v,\ g(v)\big)$$